ARULMIGU PALANIANDAVAR ARTS COLLEGE FOR WOMEN, PALANI

Department of Mathematics

Learning Resources

Title of the paper: Theory of Equations, Trigonometry and Fourier Series

Prepared By Dr.K.Meena

Ex 3 Find the fourier Series $f(x) = e^{x}$ in the interval. $0 \le x \le 2\pi$

Soln Let $f(x) = e^{x}$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{x}\,dx = \frac{1}{\pi}\left[e^{x}\right]_0^{2\pi} = \frac{1}{\pi}\left[e^{2\pi} - e^{0}\right]$$

$$a_0 = \frac{1}{\pi}\left[e^{2\pi} - 1\right] \quad (\because e^{0} = 1.)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{x}\cos nx\,dx.$$

Take $I_n = \int_0^{2\pi} e^{x}\cos nx\,dx$

$u = e^{x}$, $dv = \cos nx\,dx$

$du = e^{x}dx$, $v = \frac{\sin nx}{n}$.

$$I_n = \left(e^{x}\frac{\sin nx}{n}\right)_0^{2\pi} - \int_0^{2\pi} e^{x}\frac{\sin nx}{n}\,dx.$$

$$I_n = 0 - \frac{1}{n}\int_0^{2\pi} e^{x}\sin nx\,dx$$

$u = e^{x}$, $dv = \sin nx\,dx$

$du = e^{x}dx$, $v = \frac{-\cos nx}{n}$.

$$= -\frac{1}{n}\left[\left(-e^{x}\frac{\cos nx}{n}\right)_0^{2\pi} - \int_0^{2\pi} -e^{x}\frac{\cos nx}{n}\,dx\right]$$

$$= -\frac{1}{n}\left[\left(-\frac{e^{2\pi}\cos 2n\pi}{n} - -\frac{e^{0}\cos 0}{n}\right) + \frac{1}{n}\int_0^{2\pi} e^{x}\cos nx\,dx\right]$$

(2)

$$I_n = \frac{-1}{n}\left[\left(-\frac{e^{2\pi}}{n}+\frac{1}{n}\right)+\frac{1}{n}\int_0^{2\pi} e^{x}\cos nx\,dx\right]$$

$$= -\frac{1}{n}\left[-\frac{(e^{2\pi}-1)}{n}+\frac{1}{n}I_n\right]$$

$$\left[\begin{array}{l} \because \cos 2n\pi = 1 \\ \cos 0 = 1 \\ e^{0} = 1 \end{array}\right]$$

$$\because I_n = \int_0^{2\pi} e^{x}\cos nx\,dx$$

$$I_n = \frac{e^{2\pi}-1}{n^2} - \frac{1}{n^2}I_n$$

$$\therefore I_n + \frac{1}{n^2}I_n = \frac{e^{2\pi}-1}{n^2}$$

$$I_n\left(\frac{n^2+1}{n^2}\right) = \frac{e^{2\pi}-1}{n^2}$$

$$I_n = \frac{(e^{2\pi}-1)}{n^2} \times \frac{n^2}{(n^2+1)}$$

$$I_n = \frac{e^{2\pi}-1}{n^2+1}$$

$$\therefore a_n = \frac{1}{\pi} I_n$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi}-1}{n^2+1}\right]$$

$|||^{ly}$ $$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{x}\sin nx\,dx$$

$$= -\left[\frac{n(e^{2\pi}-1)}{\pi(n^2+1)}\right] \rightarrow \text{(Home work)}$$

$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$e^{x} = \frac{e^{2\pi}-1}{2\pi} + \sum_{n=1}^{\infty} \left[\frac{1}{\pi} \left[\frac{e^{2\pi}-1}{n^2+1} \right] \cos nx \; \frac{n(e^{2\pi}-1)}{\pi(n^2+1)} \sin nx \right]$$

$$e^{x} = \frac{e^{2\pi}-1}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{\cos nx}{n^2+1} - \left(\frac{n}{n^2+1}\right) \sin nx \right) \right].$$

Ex 4 If $f(x) = \begin{cases} -x & \text{if } -\pi \le x < 0 \\ x & \text{if } 0 \le x < \pi \end{cases}$

expand $f(x)$ as a Fourier Series in the interval $(-\pi, \pi)$.

Soln Clearly $f(-x) = f(x)$ for all $x \in (-\pi, \pi)$.

$\therefore\ f(x)$ is an even function in $(-\pi, \pi)$.

$\therefore\ f(x)$ can be expanded as a fourier Series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$. where

$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$

$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx.$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx.$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x)\,dx \quad (\text{since } f(x) \text{ is an even function})$$

$$= \frac{2}{\pi} \int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \frac{2}{\pi} \left[\frac{\pi^2}{2} - 0 \right] = \frac{2}{\pi} \times \frac{\pi^2}{2} = \pi.$$

$$a_0 = \pi.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$$ ($\because$ $f(x)$ is even function
$\cos nx$ is even function
$\therefore$ $f(x)\cos nx$ is an even function)

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\, dx.$$

$u = x,\quad dv = \cos nx\, dx$

$u' = 1 \quad v = \frac{\sin nx}{n}$

$v_1 = -\frac{\cos nx}{n^2}$

$$= \cancel{\frac{2}{\pi}\Big[-\frac{x\cos nx}{n}}$$

$\int u\,dv = uv - u'v_1.$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2}\right) - \left(\frac{0\sin 0}{n} + \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\left(0 + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi}\,\frac{(-1)^n - 1}{n^2}.$$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right].$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left((-1)^n - 1\right)\cos nx.$$

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}\cos nx.$$

Ex 5 Find a sine series for $f(x) = k$ in $0 < x < \pi$. (5)

Soln The Fourier Sine Series of $f(x)$ in $0 < x < \pi$ is given by.
$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$

$$b_n = \frac{2}{\pi}\int_0^{\pi} k \sin nx\,dx = \frac{2k}{\pi}\int_0^{\pi} \sin nx\,dx = \frac{2k}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= +\frac{2k}{n\pi}\left[-\cos n\pi + \cos 0\right]$$

$$= \frac{2k}{n\pi}\left[-(-1)^n + 1\right]$$

$$b_n = \frac{2k}{n\pi}\left((-1)^{n+1} + 1\right). = \begin{cases} \frac{4k}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2k}{n\pi}\left((-1)^{n+1} + 1\right)\sin nx.$$

$$= \frac{2k}{\pi}\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n}\sin nx\right]$$

$$= \frac{2k}{\pi}\left[\frac{2}{1}\sin x + 0 + \frac{2}{3}\sin 3x + 0 + \frac{2}{5}\sin 5x + \cdots\right].$$

$$f(x) = \frac{4k}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$

$$k = \frac{4k}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

Ex6 Find the Fourier (i) Cosine series (ii) Sine Series for the function $f(x) = \pi - x$ in $(0, \pi)$. (2)

Soln

(i) Let $f(x) = \pi - x$.

The Fourier Cosine series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$, $\quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\,dx = \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\pi\times\pi - \frac{\pi^2}{2}\right) - 0\right]$$

$$= \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right] = \frac{2}{\pi}\times\frac{\pi^2}{2} = \pi$$

$a_0 = \pi$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx$$

$u = \pi - x$, $dv = \cos nx\,dx$

$u' = -1$ $\quad v = \frac{\sin nx}{n}$

$v_1 = -\frac{\cos nx}{n^2}$

$$= \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 - \frac{\cos n\pi}{n^2}\right) - \left(0 - \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos n\pi}{n^2} + \frac{\cos 0}{n^2}\right] = \frac{2}{\pi n^2}\left[-\cos n\pi + \cos 0\right]$$

$$a_n = \frac{2}{\pi n^2}\left[-(-1)^n + 1\right]$$

$$a_n = \frac{2}{\pi n^2}\left((-1)^{n+1} + 1\right).$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left((-1)^{n+1} + 1\right)\cos nx$$

$$\pi - x = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n^2}\cos nx.$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{2\cos x}{1^2} + 0 + \frac{2}{3^2}\cos 3x + 0 + \frac{2}{5^2}\cos 5x + \cdots\right]$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

$$\pi - x = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}.$$

(ii) Let $f(x) = \pi - x$. The fourier sine series of $f(x)$ is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$u = \pi - x$, $dv = \sin nx\,dx$

$u' = -1$, $v = \frac{-\cos nx}{n}$

$v_1 = \frac{-\sin nx}{n^2}$

$$= \frac{2}{\pi}\left[\frac{-(\pi - x)\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(0-0) - \left(\frac{-(\pi - 0)\cos 0}{n} - 0\right)\right]$$

$\because \sin n\pi = 0$, $\sin 0 = 0$, $\cos 0 = 1$

$$= \frac{2}{\pi}\left[\frac{\pi}{n}\right] = \frac{2}{n}.$$

$$\therefore \pi - x = \sum_{n=1}^{\infty} \frac{2}{n}\sin nx \Rightarrow \pi - x = 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

Ex 4 show that $0 < x < \pi$

$$\pi - x = \frac{\pi}{2} + \frac{\sin 2x}{1} + \frac{\sin 4x}{2} + \frac{\sin 6x}{3} + \cdots$$

Soln Consider $f(x) = \pi - x - \frac{\pi}{2} = \frac{\pi}{2} - x$.

We now find the sine series of $f(x)$.

$$b_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)\sin nx \, dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2} - x\right)\left(-\frac{\cos nx}{n}\right) - \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$u = \frac{\pi}{2} - x$, $dv = \sin nx \, dx$

$u' = -1$, $v = \frac{-\cos nx}{n}$

$v_1 = \frac{-\sin nx}{n^2}$

$$= \frac{2}{\pi}\left[\left\{\left(\frac{\pi}{2} - \pi\right)\left(-\frac{\cos n\pi}{n}\right) - 0\right\} - \left\{\left(\frac{\pi}{2} - 0\right)\left(-\frac{\cos 0}{n}\right) - 0\right\}\right]$$

$$= \frac{2}{\pi}\left[\left(-\frac{\pi}{2}\right)\left(\frac{-(-1)^n}{n}\right) - \left(\left(\frac{\pi}{2}\right)\left(-\frac{1}{n}\right)\right)\right]$$

$\because \cos n\pi = (-1)^n$

$\cos 0 = 1$

$\sin n\pi = 0$

$\sin 0 = 0$

$$= \frac{2}{\pi}\left[+\frac{\pi}{2n}(-1)^n + \frac{\pi}{2n}\right]$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{2n}\left[(-1)^n + 1\right]$$

$$b_n = \frac{1}{n}\left[(-1)^n + 1\right]$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\frac{\pi}{2} - x = \sum_{n=1}^{\infty} \frac{1}{n}\left[(-1)^n + 1\right]\sin nx.$$

$$= 0 + \frac{2\sin 2x}{2} + 0 + \frac{2}{4}\sin 4x + 0 + \frac{2\sin 6x}{6} + \cdots$$

$$\therefore \frac{\pi}{2} - x = \frac{2}{2}\left[\frac{\sin 2x}{1} + \frac{\sin 4x}{2} + \frac{\sin 6x}{3} + \cdots\right]$$

$$\therefore \pi - x - \frac{\pi}{2} = \frac{\sin 2x}{1} + \frac{\sin 4x}{2} + \frac{\sin 6x}{3} + \cdots$$

$$\boxed{\therefore \pi - x = \frac{\pi}{2} + \frac{\sin 2x}{1} + \frac{\sin 4x}{2} + \frac{\sin 6x}{3} + \cdots}$$

Excercises Problem

Find the Fourier Series to represent $f(x)$ in $(-\pi, \pi)$ if

(i) $f(x) = \begin{cases} -1 & \text{in } -\pi < x < 0 \\ 1 & \text{in } 0 \leq x < \pi \end{cases}$

(ii) $f(x) = \begin{cases} 1 & \text{in } -\pi < x \leq 0 \\ -2 & \text{in } 0 < x \leq \pi \end{cases}$

(iii) $f(x) = \begin{cases} 0 & \text{in } -\pi < x \leq 0 \\ \frac{1}{4}\pi x & \text{in } 0 < x < \pi \end{cases}$

(ii) Soln $\quad f(x) = \begin{cases} -1 & \text{in } -\pi < x < 0 \\ 1 & \text{in } 0 \le x < \pi \end{cases}$ (10)

fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)dx + \int_{0}^{\pi} 1\, dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{-\pi}^{0} dx + \int_{0}^{\pi} dx\right] = \frac{1}{\pi}\left[-(x)_{-\pi}^{0} + (x)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-[0-(-\pi)] + (\pi - 0)\right]$$

$$= \frac{1}{\pi}\left[-\pi + \pi\right] = \frac{1}{\pi}(0).$$

$a_0 = 0$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\, dx + \int_{0}^{\pi} f(x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\cos nx\, dx + \int_{0}^{\pi} \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{\sin nx}{n}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-(0) + 0\right] = 0.$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

(II)

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-1)\sin nx\,dx + \int_{0}^{\pi} 1\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\left(-\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(\frac{-\cos nx}{n}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\left[\cos 0 - \cos n(-\pi)\right] - \frac{1}{n}\left[\cos n\pi - \cos 0\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\left[1 - \cos n\pi\right] - \frac{1}{n}\left[\cos n\pi - 1\right]\right]$$

$$= \frac{1}{n\pi}\left[1 - \cos n\pi - \cos n\pi + 1\right]$$

$$= \frac{1}{n\pi}\left[2 - 2\cos n\pi\right] = \frac{2}{n\pi}\left[1 - \cos n\pi\right]$$

$$b_n = \frac{2}{n\pi}\left[1 - (-1)^n\right] \text{ (or) } = \frac{2}{n\pi}\left[1 + (-1)^{n+1}\right]$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 + (-1)^{n+1}\right)\sin nx.$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n}\sin nx$$

$$= \frac{2}{\pi}\left[\frac{2}{1}\sin x + 0 + \frac{2}{3}\sin 3x + 0 + \frac{2}{5}\sin 5x + \cdots\right]$$

$$f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right].$$

Expansion of $\sin\theta$, $\cos\theta$, $\tan\theta$ in Powers of θ.

Thm When θ is expressed in radians.

(i) $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$

(ii) $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$

(iii) $\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \cdots$

Proof (i) Let $f(\theta) = \sin\theta$

The Taylor series expansion of f about the origin is given by

$$f(\theta) = f(0) + f'(0)\frac{\theta}{1!} + f''(0)\frac{\theta^2}{2!} + f'''(0)\frac{\theta^3}{3!} + \cdots$$

now $f(\theta) = \sin\theta$ $\quad f'(\theta) = \cos\theta \quad f'(0) = 1$

$f(0) = 0$ $\quad f''(\theta) = -\sin\theta \quad f''(0) = 0$

$f'''(\theta) = -\cos\theta \quad f'''(0) = -1$

$\vdots \qquad \vdots$

$$\therefore \sin\theta = 0 + 1\times\frac{\theta}{1!} + 0 + (-1)\frac{\theta^3}{3!} + 0 + \frac{\theta^5}{5!} + \cdots$$

$\cancel{\sin\theta = \theta + \frac{\theta^3}{3!}}$.

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

(ii) $f(\theta) = \cos\theta$ $\qquad f'(\theta) = -\sin\theta$

$f(0) = 1$, $\quad f'$, $\quad f''(\theta) = -\cos\theta$

$f'(0) = 0$ $\qquad f'''(\theta) = \sin\theta$

$f''(0) = -1$ $\qquad f^{iv}(\theta) = \cos\theta$

$f'''(0) = 0$ $\qquad f^{v}(\theta) = -\sin\theta$

$f^{iv}(0) = 1$

$f^{v}(0) = 0$ etc.

$$\cos\theta = 1 + 0 - \frac{\theta^2}{2!} + 0 + \frac{\theta^4}{4!} - \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

(iii)

Let $f(\theta) = \tan\theta$. $f(0) = 0$.

$f'(\theta) = \sec^2\theta$. $f'(0) = 1$.

$f''(\theta) = 2\sec^2\theta\tan\theta$ $\quad f''(0) = 0$.

$= 2(1+\tan^2\theta)\tan\theta = 2(\tan\theta + \tan^3\theta)$.

$f'''(\theta) = 2[\sec^2\theta + 3\tan^2\theta \times \sec^2\theta] = 2[\sec^2\theta + 3\tan^2\theta(1+\tan^2\theta)]$

$= 2[\sec^2\theta + 3\tan^2\theta + 3\tan^4\theta]$

$f^{iv}(\theta) = 2[2\sec\theta\sec\theta\tan\theta + 6\tan\theta\sec^2\theta + 12\tan^3\theta\sec^2\theta]$

$f'''(0) = 2$.

$f^{iv}(\theta) = 2[8\sec^2\theta\tan\theta + 12\sec^2\theta\tan^3\theta]$

$f^{iv}(\theta) = 2[0] = 0$

$f^{iv}(\theta) = 2[8(1+\tan^2\theta)\tan\theta + 12(1+\tan^2\theta)\tan^3\theta]$

$= 2[8\tan\theta + 8\tan^3\theta + 12\tan^3\theta + 12\tan^5\theta]$

$= 2[12\tan^5\theta + 20\tan^3\theta + 8\tan\theta]$

$f^{iv}(\theta) = 8[3\tan^5\theta + 5\tan^3\theta + 2\tan\theta]$

$f^{v}(\theta) = 8[15\tan^4\theta\sec^2\theta + 15\tan^2\theta\sec^2\theta + 2\sec^2\theta]$

$f^{(v)}(0) = 8(2) = 16$. $\left(\because \begin{matrix}\sec 0 = 1 \\ \tan 0 = 0\end{matrix}\right)$

$$\therefore \tan\theta = 0 + \frac{\theta}{1!} + 0 + \frac{2\theta^3}{3!} + 0 + 16\frac{\theta^5}{5!} + \cdots$$

$$= \theta + \frac{2\times\theta^3}{1\times2\times3} + \frac{16}{1\times2\times3\times4\times5}\theta^5 + \cdots$$

$$= \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \cdots$$

Ex Find approximately the value of θ in radians

If $\frac{\sin\theta}{\theta} = \frac{863}{864}$.

VB

Soln

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots$$

$$\frac{863}{864} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots$$

$$1 - \frac{1}{864} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots$$

$$\frac{1}{864} = \frac{\theta^2}{3!} - \frac{\theta^4}{5!} + \cdots$$

$\frac{\theta^2}{3!} \approx \frac{1}{864}$ (neglecting higher powers of θ.)

$\theta^2 \approx \frac{6}{864} = \frac{1}{144} \Rightarrow \theta^2 \approx \frac{1}{144} \Rightarrow \theta \approx \frac{1}{12}$ radians. (approximate)

Ex 2 If $\frac{\tan\theta}{\theta} = \frac{2524}{2523}$ show that θ is approximately equal to $1^\circ 58'$.

(14)

Soln $\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \cdots$

$$\frac{\tan\theta}{\theta} = 1 + \frac{\theta^2}{3} + \frac{2}{15}\theta^4 + \cdots$$

$$\frac{2524}{2523} = 1 + \frac{\theta^2}{3} + \frac{2}{15}\theta^4 + \cdots$$

$$1 + \frac{1}{2523} = 1 + \frac{\theta^2}{3} + \frac{2}{15}\theta^4 + \cdots$$

$$\frac{1}{2523} = \frac{\theta^2}{3} + \frac{2}{15}\theta^4 + \cdots$$

$\frac{\theta^2}{3} \approx \frac{1}{2523}$ neglecting higher power of θ.

π radians $= 180$ degree

$$\theta^2 \approx \frac{3}{2523} = \frac{1}{841}.$$

1 radian $= \frac{180}{\pi}$ degree.

$\theta \approx \frac{1}{29}$ radians

1 radian $= \frac{180}{\frac{22}{7}}$

$= \frac{1}{29} \times 57.27$ degrees approximately.

1 radian $= \frac{180 \times 7}{22}$

$\theta \approx 1^\circ 58'$ approximately.

$= \frac{1260}{22} = 57.27272$

$\boxed{1 \text{ radian} \approx 57.27 \text{ degree}}$

$$\frac{1}{29} \times 57.27 = 1.97482759.$$

$$= 1^\circ 58' 29''$$

0.4896554×60 seconds

$= 29.37$ second

$1' = 60''$ seconds

$1^\circ = 60'$ minutes

$0.97482759 \times 60'$

$= 58'.4896554\ldots$

Ex 4 Solve approximately $\cos(\frac{\pi}{3}+\theta) = 0.49$.

15

Soln

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(\frac{\pi}{3}+\theta) = 0.49$

$\cos\frac{\pi}{3}\cos\theta - \sin\frac{\pi}{3}\sin\theta = 0.49$

$\frac{1}{2}\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}\cdots\right) - \frac{\sqrt{3}}{2}\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}\right) = 0.49.$

$\frac{1}{2} - \frac{\sqrt{3}}{2}\theta \simeq 0.49$ (neglecting higher powers of θ)

$\frac{\sqrt{3}\theta}{2} = \frac{1}{2} - 0.49 \left(= 0.50 - 0.49 = 0.01 = \frac{1}{100}\right).$

$\frac{\sqrt{3}\theta}{2} = \frac{1}{100}$

$\theta = \frac{1}{50\sqrt{3}} \Rightarrow \theta = \frac{3}{3\times 50\sqrt{3}} = \frac{\sqrt{3}\sqrt{3}}{150\sqrt{3}} = \frac{\sqrt{3}}{150}.$

$\theta = \frac{\sqrt{3}}{150} = \frac{1.732}{150} = 0.0115$ radian.

$\theta = 0.0115 \times 57.27$ degree

$= 0.658605$ degree

$= 0.658605 \times 60$ minutes. approx

$= 39.51.$

$\theta \simeq 40$ minutes // (approximately)

Ex Evaluate $\sin 3^\circ$ correct to three places of decimals

Soln $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$ (θ radians)

π radian $= 180^\circ$ $\quad$ 1 radian $= \frac{180}{\pi}$ degree

$1^\circ = \frac{\pi}{180}$ radian.

(16)

$3^\circ = \frac{3\pi}{180}$ radian $= \frac{\pi}{60}$ radian

$\therefore \sin 3^\circ = \sin\left(\frac{\pi}{60}\right) = \frac{\pi}{60} - \frac{1}{3!}\left(\frac{\pi}{60}\right) + \cdots$

$\sin 3^\circ \approx \frac{\pi}{60}$ (neglecting higher powers)

$= \frac{22}{7} \times \frac{1}{60} \approx 0.052$

$\sin 3^\circ \approx 0.052.$

Ex If θ is small Prove that $\theta \cot\theta = 1 - \frac{\theta^2}{3} - \frac{\theta^4}{45}$ (approx)

Soln $\theta \cot\theta = \frac{\theta}{\tan\theta}$

$= \frac{\theta}{\theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5}$ (neglecting higher powers of θ)

$= \frac{1}{1 + \frac{\theta^2}{3} + \frac{2}{15}\theta^4}$ $\because (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

$= \left(1 + \left(\frac{\theta^2}{3} + \frac{2}{15}\theta^4\right)\right)^{-1}$

$= 1 - \left(\frac{\theta^2}{3} + \frac{2}{15}\theta^4\right) + \left(\frac{\theta^2}{3} + \frac{2}{15}\theta^4\right)^2 - \cdots$

$= 1 - \frac{\theta^2}{3} - \frac{2}{15}\theta^4 + \frac{\theta^4}{9}$ (neglecting higher powers of θ)

$= 1 - \frac{\theta^2}{3} - \left(\frac{2}{15} - \frac{1}{9}\right)\theta^4$

$\theta \cot\theta = 1 - \frac{\theta^2}{3} - \frac{\theta^4}{45}$ (approximately)

Ex Prove that when θ is small. $\frac{1}{6}\sin^3\theta = \frac{\theta^3}{3!} - (1+3^2)\frac{\theta^5}{5!} + (1+3^2+3^4)\frac{\theta^7}{7} \cdots$

Sdn

n is odd, $\sin^n\theta = \frac{1}{2^{n-1}(-1)^{\frac{n-1}{2}}}\left[\sin n\theta - nC_1 \sin(n-2)\theta + nC_2 \sin(n-4)\theta - \cdots + (-1)^{\frac{n-1}{2}} nC_{\frac{n-1}{2}} \sin\theta\right]$

$\sin^3\theta = \frac{1}{2^2(-1)^1}\left[\sin 3\theta - 3C_1 \sin\theta\right]$

$= \frac{-1}{2^2}\left[3\theta - \frac{(3\theta)^3}{3!} + \frac{(3\theta)^5}{5!} - \cdots - 3\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right]\right]$

(17)

$$\sin^3\theta = \frac{1}{4}\Big[-3\theta + \frac{3^3\theta^3}{3!} - \frac{3^5\theta^5}{5!} + \frac{3^7\theta^7}{7!} - \cdots$$
$$+3\theta - \frac{3\theta^3}{3!} + \frac{3\theta^5}{5!} - \frac{3\theta^7}{7!} + \cdots\Big]$$

$$= \frac{1}{4}\left[-\frac{\theta^3}{3!}(3-3^3) + \frac{\theta^5}{5!}(3-3^5) - \frac{\theta^7}{7!}(3-3^7) + \cdots\right]$$

$$\sin^3\theta = \frac{3}{4}\left[(3^2-1)\frac{\theta^3}{3!} - (3^4-1)\frac{\theta^5}{5!} + (3^6-1)\frac{\theta^7}{7!} - \cdots\right]$$

$$\sin^3\theta = \frac{3}{4}\times(3^2-1)\left[\frac{\theta^3}{3!} - (3^2+1)\frac{\theta^5}{5!} + (3^4+3^2+1)\frac{\theta^7}{7!} - \cdots\right]$$

$$\frac{\sin^3\theta}{6} = \frac{\theta^3}{3!} - (3^2+1)\frac{\theta^5}{5!} + (3^4+3^2+1)\frac{\theta^7}{7!} - \cdots$$

$$\because (3^2-1)\times\frac{3}{4} = \frac{8\times 3}{4} = 6.$$

$$3^4-1 = (3^2)^2 - 1^2 = (3^2-1)(3^2+1)$$

$$a^2-b^2 = (a-b)(a+b)$$

$$3^6-1 = (3^2)^3 - 1^3 \quad (\because a^3-b^3 = (a-b)(a^2+ab+b^2))$$

$$3^6-1 = (3^2-1)(3^4+3^2+1).$$

Ex Show that $\lim\limits_{x\to 0} \dfrac{3\sin x - \sin 3x}{x - \sin x} = 24.$

Soln $\lim\limits_{x\to 0} \dfrac{3\sin x - \sin 3x}{x-\sin x} = \lim\limits_{x\to 0} \dfrac{3\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - \left(3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \cdots\right)}{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}$

$$= \lim_{x\to 0} \frac{3x - 3x - \frac{3x^3}{3!} + \frac{3^3x^3}{3!} + \frac{3x^5}{5!} - \frac{3^5x^5}{5!} + \cdots}{x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots}$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{3!}\left[3^3-3\right] + \frac{x^5}{5!}\left[3-3^5\right] + \cdots}{\frac{x^3}{3!} - \frac{x^5}{5!} + \cdots}$$

$$= \lim_{x\to 0} \frac{x^3\left[\frac{3^3-3}{3!} + \frac{x^2}{5!}(3-3^5) + \cdots\right]}{x^3\left(\frac{1}{3!} - \frac{x^2}{5!} + \cdots\right)}$$

(18)

$$= \lim_{x \to 0} \left(\frac{\frac{3^3 - 3}{3!} - \frac{x^2}{5!}(3 - 3^5) + \cdots}{\frac{1}{3!} - \frac{x^2}{5!} + \cdots} \right)$$

$$= \frac{\frac{3^3 - 3}{3!}}{\frac{1}{3!}} = \frac{3^3 - 3}{3!} \times \frac{3!}{1} = 3^3 - 3.$$

$$= 27 - 3 = 24.$$

Ex Show that $\lim_{x \to 0} \left(\frac{\cos^2 ax - \cos^2 bx}{1 - \cos cx} \right) = \frac{2(b^2 - a^2)}{c^2}$

LHS $\Rightarrow$

$$\lim_{x \to 0} \frac{\cos^2 ax - \cos^2 bx}{1 - \cos cx} = \lim_{x \to 0} \frac{\left(1 - \frac{a^2x^2}{2!} + \frac{a^4x^4}{4!} \cdots\right)^2 - \left(1 - \frac{b^2x^2}{2!} + \frac{b^4x^4}{4!} - \cdots\right)^2}{1 - \left(1 - \frac{c^2x^2}{2!} + \frac{c^4x^4}{4!} - \cdots\right)}$$

$$= \lim_{x \to 0} \frac{1 + \frac{a^4x^4}{4} - 2\frac{a^2x^2}{2!} + \cdots - \left(1 + \frac{b^4x^4}{4} - 2\frac{b^2x^2}{2!} + \cdots\right)}{\frac{c^2x^2}{2!} - \frac{c^4x^4}{4!} + \cdots}$$

$$= \lim_{x \to 0} \frac{\frac{2b^2x^2}{2!} - \frac{2a^2x^2}{2!} + \frac{a^4x^4}{4} - \frac{b^4x^4}{4} + \cdots}{\frac{c^2x^2}{2!} - \frac{c^4x^4}{4!} + \cdots}$$

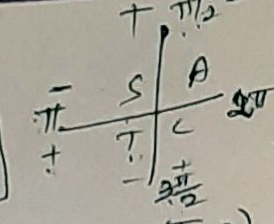
$$= \lim_{x \to 0} \frac{x^2\left[\frac{2b^2}{2} - \frac{2a^2}{2} + \frac{a^4x^2}{4} - \frac{b^4x^2}{4} + \cdots\right]}{x^2\left(\frac{c^2}{2} - \frac{c^4x^2}{4!} + \cdots\right)}.$$

$$= \lim_{x \to 0} \frac{b^2 - a^2 + \frac{a^4x^2}{4} - \frac{b^4x^2}{4} + \cdots}{\frac{c^2}{2} - \frac{c^4x^2}{4!} + \cdots} = \frac{b^2 - a^2}{\frac{c^2}{2}} = \frac{2(b^2 - a^2)}{c^2}$$

$$= \frac{2(b^2 - a^2)}{c^2} = RHS.$$

(19) **Ex** Evaluate $\lim\limits_{x \to \pi/2} \dfrac{\sin x + \cos 2x}{\cos^2 x}$.

Put $x = \theta + \pi/2$. As $x \to \pi/2$ we notice $\theta \to 0$.



$$\lim_{x \to \pi/2} \frac{\sin x + \cos 2x}{\cos^2 x} = \lim_{\theta \to 0} \frac{\cos\theta - \cos 2\theta}{\sin^2\theta}$$

$\cos(\theta + \frac{\pi}{2}) = \cos(\frac{\pi}{2} + \theta)$

$\cos(\frac{\pi}{2} + \theta) = -\sin\theta$

$\sin(\frac{\pi}{2} + \theta) = \cos\theta$.

$\cos 2(\frac{\pi}{2} + \theta)$

$= \cos(\pi + 2\theta)$

$= -\cos 2\theta$

3rd quadrant tan & cot +ve. other -ve.

$$= \lim_{\theta \to 0} \frac{\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right] - \left[1 - \frac{4\theta^2}{2!} + \frac{16\theta^4}{4!} - \cdots\right]}{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)^2}$$

$$= \lim_{\theta \to 0} \frac{\cancel{1} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots - \cancel{1} + \frac{4\theta^2}{2!} - \frac{16\theta^4}{4!} + \cdots}{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)^2}$$

$$= \lim_{\theta \to 0} \frac{\cancel{\theta^2}\left[\left(-\frac{1}{2!} + \frac{4}{2!}\right) + \frac{\theta^2}{4!} - \frac{16\theta^2}{4!} - \cdots\right]}{\cancel{\theta^2}\left(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots\right)^2}$$

$$= \frac{-\frac{1}{2!} + \frac{4}{2!}}{1} = -\frac{1}{2} + \frac{4}{2} = \frac{-1+4}{2} = \frac{3}{2}.$$

$$= \frac{3}{2}$$

Home work I Find the value of θ when

(1) $\dfrac{\sin\theta}{\theta} = \dfrac{2165}{2166}$. (2) $\dfrac{\sin\theta}{\theta} = \dfrac{19493}{19494}$.

II (3) $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$, (4) $\lim\limits_{x \to 0} \dfrac{\sin 2x - 2\sin x}{x^3}$.

① Expression for $\sin n\theta$, $\cos n\theta$ and $\tan n\theta$.

Thm For any positive integer n.

(i) $\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + \cdots$

(ii) $\sin n\theta = n\cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + \cdots$

Proof By De Moivre's Theorem.

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

$$= \cos^n\theta + {}^nC_1 \cos^{n-1}\theta\,(i\sin\theta) + {}^nC_2 \cos^{n-2}\theta\,(i\sin\theta)^2$$
$$+ {}^nC_3 \cos^{n-3}\theta\,(i\sin\theta)^3 + {}^nC_4 \cos^{n-4}\theta\,(i\sin\theta)^4$$
$$+ \cdots$$

$$= \cos^n\theta + i\,n\cos^{n-1}\theta\sin\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta$$
$$- i\,{}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta.$$

$$\cos n\theta + i\sin n\theta.\ - i\,{}^nC_5\cos^{n-5}\theta\sin^5\theta + \cdots$$

$\because {}^nC_1 = n,$

$\because (a+b)^n = a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + {}^nC_3 a^{n-3} b^3 + \cdots + {}^nC_{n-1}\, a\, b^{n-1} + {}^nC_n b^n.$

$i^2 = -1,\quad i^3 = -i,\quad i^4 = 1.$ etc.

Equating real and imaginary parts. We get ① & (ii). respectively.

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \cdots$$

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \cdots$$

Corollary For any +ve integer n.

$$\tan n\theta = \frac{nc_1 \tan\theta - nc_3 \tan^3\theta + \cdots}{1 - nc_2 \tan^2\theta + nc_4 \tan^4\theta - \cdots} \qquad (2)$$

Proof

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}.$$

$$= \frac{nc_1 \cos^{n-1}\theta \sin\theta - nc_3 \cos^{n-3}\theta \sin^3\theta + \cdots}{\cos^n\theta - nc_2 \cos^{n-2}\theta \sin^2\theta + nc_4 \cos^{n-4}\theta \sin^4\theta - \cdots}$$

$$= \frac{\cancel{\cos^n\theta}}{\cancel{\cos^n\theta}} \left[\frac{nc_1 \frac{\sin\theta}{\cos\theta} - nc_3 \frac{\sin^3\theta}{\cos^3\theta} + nc_5 \frac{\sin^5\theta}{\cos^5\theta} - \cdots}{1 - nc_2 \frac{\sin^2\theta}{\cos^2\theta} + nc_4 \frac{\sin^4\theta}{\cos^4\theta} - \cdots} \right]$$

$$\tan n\theta = \frac{nc_1 \tan\theta - nc_3 \tan^3\theta + nc_5 \tan^5\theta - \cdots}{1 - nc_2 \tan^2\theta + nc_4 \tan^4\theta - \cdots}$$

Ex Expand $\sin 7\theta$ in powers of $\cos\theta$ and $\sin\theta$.

Hence Prove that $\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$.

Soln

~~cosx~~.

$$\sin n\theta = nc_1 \cos^{n-1}\theta \sin\theta - nc_3 \cos^{n-3}\theta \sin^3\theta.$$
$$+ nc_5 \cos^{n-5}\theta \sin^5\theta - \cdots$$

$$\sin 7\theta = 7c_1 \cos^6\theta \sin\theta - 7c_3 \cos^4\theta \sin^3\theta$$
$$+ 7c_5 \cos^2\theta \sin^5\theta - 7c_7 \cos^0\theta \sin^7\theta$$

$$= 7\cos^6\theta \sin\theta - \frac{7\times6\times5}{1\times2\times3} \cos^5\theta \sin^2\theta$$
$$+ \frac{7\times6\times5\times4\times3}{1\times2\times3\times4\times5} \cdot \cos^2\theta \sin^5\theta - \sin^7\theta.$$

$$= 7\cos^6\theta\sin\theta - 35\cos^5\theta\sin^2\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$

$$\boxed{\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^5\theta\sin^2\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta}$$

$$\frac{\sin 7\theta}{\sin\theta} = 7\cos^6\theta - 35\cos^5\theta\sin\theta + 21\cos^2\theta\sin^4\theta - \sin^6\theta.$$

$$7C_5 = 7C_2 = \frac{7\times 6}{1\times 2} = 21.$$

$$7C_3 = \frac{7\times 6\times 5}{1\times 2\times 3}$$

$nC_n = 1.$

$nC_r = nC_{n-r}$

$nC_0 = 1$

$nC_1 = n.$

(3)

$$\frac{\sin 7\theta}{\sin\theta} = 7(1-\sin^2\theta)^3 - 35(1-\sin^2\theta)^2\sin^2\theta + 21(1-\sin^2\theta)\sin^4\theta - \sin^6\theta \quad [\because \cos^2\theta = 1-\sin^2\theta]$$

$$= 7[1-\sin^6\theta - 3\sin^2\theta + 3\sin^4\theta] - 35[1-2\sin^2\theta+\sin^4\theta]\sin^2\theta + 21(1-\sin^2\theta)\sin^4\theta - \sin^6\theta$$

$$= 7 - 7\sin^6\theta - 21\sin^2\theta + 21\sin^4\theta - 35\sin^2\theta + 70\sin^4\theta - 35\sin^6\theta + 21\sin^4\theta - 21\sin^6\theta - \sin^6\theta$$

$$= 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta.$$

Ex Prove that $\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1.$

Solution

$$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta - nC_6\cos^{n-6}\theta\sin^6\theta + \cdots$$

$$\cos 8\theta = \cos^8\theta - 8C_2\cos^6\theta\sin^2\theta + 8C_4\cos^4\theta\sin^4\theta - 8C_6\cos^2\theta\sin^6\theta + 8C_8\cos^0\theta\sin^8\theta.$$

(5)

$= \cos^8\theta - 28\cos^6\theta\sin^2\theta + 70\cos^4\theta\sin^4\theta$
$\quad - 28\cos^2\theta\sin^6\theta + \sin^8\theta$

$= \cos^8\theta - 28\cos^6\theta(1-\cos^2\theta)$
$\quad + 70\cos^4\theta(1-\cos^2\theta)^2$
$\quad - 28\cos^2\theta(1-\cos^2\theta)^3$
$\quad + (1-\cos^2\theta)^4.$

$= \cos^8\theta - 28\cos^6\theta(1-\cos^2\theta)$
$\quad + 70\cos^4\theta(1-2\cos^2\theta+\cos^4\theta)$
$\quad - 28\cos^2\theta(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)$
$\quad + (1-4\cos^2\theta+6\cos^4\theta-4\cos^6\theta+\cos^8\theta).$

$= \cos^8\theta - 28\cos^6\theta + 28\cos^8\theta$
$\quad + 70\cos^4\theta - 140\cos^6\theta + 70\cos^8\theta$
$\quad - 28\cos^2\theta + 84\cos^4\theta - 84\cos^6\theta + 28\cos^8\theta$
$\quad + 1 - 4\cos^2\theta + 6\cos^4\theta - 4\cos^6\theta + \cos^8\theta$

$= 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1$ //.

$8C_1 = 8$

$8C_2 = \frac{8\times 7}{1\times 2}$

$8C_2 = 28$

$8C_3 = \frac{8\times 7\times 6}{1\times 2\times 3}$

$= 56$

$8C_4 = \frac{8\times 7\times 6\times 5}{1\times 2\times 3\times 4}$

$= 70$

$8C_5 = 8C_3$

$8C_6 = 8C_2$

$8C_0 = 1$

$8C_8 = 1.$

$(a-b)^4$

$4C_2 = \frac{4\times 3}{1\times 2}$

$= 6$

$4C_3 = 4C_1$

$= 4.$

$(a-b)^4 = a^4 - 4C_1 a^3 b + 4C_2 a^2 b^2 - 4C_3 a b^3 + 4C_4 b^4.$

$= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$

~~H~~ome work

Prove that (i) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(ii) $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

(iii) $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$.

(3)

Expression for $\sin^n\theta$ and $\cos^n\theta$.

Thm When n is a positive integer.

$$\cos^n\theta = \frac{1}{2^{n-1}}\left[\cos n\theta + nc_1 \cos(n-2)\theta + nc_2\cos(n-4)\theta + \dots\right]$$

Proof Let $x = \cos\theta + i\sin\theta$

$\therefore \frac{1}{x} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta.$

and $x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$\frac{1}{x^n} = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta.$

$x + \frac{1}{x} = 2\cos\theta.$

$x^n + \frac{1}{x^n} = 2\cos n\theta.$

$2\cos\theta = x + \frac{1}{x}$

$(2\cos\theta)^n = (x + \frac{1}{x})^n$

$\therefore$ ~~$2\cos\theta$~~

$\therefore (2\cos\theta)^n = (x+\frac{1}{x})^n$

$= x^n + nc_1 x^{n-1}(\frac{1}{x}) + nc_2 x^{n-2}(\frac{1}{x})^2 + nc_3 x^{n-3}(\frac{1}{x})^3 + \dots$

$+ nc_{n-1} \cancel{x} (\frac{1}{x})^{n-1} + \frac{1}{x^n}.$

$2^n\cos^n\theta = x^n + nc_1 x^{n-2} + nc_2 x^{n-4} + nc_3 x^{n-6} + \dots + nc_{n-1}\frac{1}{x^{n-2}} + \frac{1}{x^n}$

[Since $nc_{n-1} = nc_1$, $nc_{n-2} = nc_2, \dots$

$\therefore nc_{n-r} = nc_r$.]

$2^n\cos^n\theta = x^n + \frac{1}{x^n} + nc_1\left[x^{n-2} + \frac{1}{x^{n-2}}\right] + \dots$

$= 2\cos n\theta + nc_1 (2\cos(n-2)\theta] + nc_3 (2\cos(n-4)\theta) + \dots$

(6)

$$\cos^n\theta = \frac{1}{2^{n-1}}\left[\cos n\theta + nC_1 \cos(n-2)\theta + nC_2 \cos(n-4)\theta + \cdots \right.$$
$$\left. \cdots\right]$$

Thm When n is a positive integer

$$\sin^n\theta = \begin{cases} \dfrac{1}{(-1)^{n/2}\, 2^{n-1}}\left[\cos n\theta - nC_1\cos(n-2)\theta + nC_2\cos(n-4)\theta - \cdots \right. \\ \qquad \left. \cdots + \dfrac{(-1)^{n/2}}{2}\, nC_{n/2}\right] & \text{If } n \text{ is even} \\ \dfrac{1}{(-1)^{\frac{n-1}{2}}\, 2^{n-1}}\left[\sin n\theta - nC_1 \sin(n-2)\theta + nC_2 \sin(n-4)\theta \right. \\ \qquad \left. - \cdots + (-1)^{\frac{n-1}{2}}\, nC_{\frac{n-1}{2}} \sin\theta\right] & \text{If } n \text{ is odd.} \end{cases}$$

Proof Let $x = \cos\theta + i\sin\theta$, $\quad \frac{1}{x} = \cos\theta - i\sin\theta$

$x^n = \cos n\theta + i\sin n\theta \qquad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$x - \frac{1}{x} = 2i\sin\theta, \qquad x^n + \frac{1}{x^n} = 2\cos n\theta$

$x + \frac{1}{x} = 2\cos\theta \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$

$$\therefore (2i\sin\theta)^n = \left(x - \frac{1}{x}\right)^n$$

$$2^n i^n \sin^n\theta = x^n - nC_1\, x^{n-1}\left(\frac{1}{x}\right) + nC_2\, x^{n-2}\left(\frac{1}{x}\right)^2 - nC_3\, x^{n-3}\left(\frac{1}{x}\right)^3 + \cdots$$
$$+ (-1)^{n-3}\, nC_{n-3}\, x^3 \left(\frac{1}{x}\right)^{n-3} + (-1)^{n-2}\, nC_{n-2}\, x^2 \left(\frac{1}{x}\right)^{n-2} + (-1)^{n-1}\, nC_{n-1}\, x\left(\frac{1}{x}\right) + \frac{(-1)^n}{x^n}$$

$$2^n i^n \sin^n\theta = x^n - nC_1\, x^{n-2} + nC_2\, x^{n-4} - nC_3\, x^{n-6} + \cdots$$
$$+ (-1)^{n-3} \cdot nC_{n-3} \frac{1}{x^{n-6}} + (-1)^{n-2}\, nC_{n-2} \frac{1}{x^{n-4}}$$
$$+ (-1)^{n-1}\, nC_{n-1} \frac{1}{x^{n-2}} + (-1)^n \frac{1}{x^n} \quad \text{—— (1)}$$

$nC_{n-2} = nC_2$

$nC_{n-3} = nC_3$

$nC_{n-1} = nC_1$

Step(i) n is even.

$\therefore$ The no. of terms in the RHS is odd and hence the last term is +ve $\because (-1)^{\text{even}} \to$ positive)

Middle term is independent of x. and it is $(-1)^{n/2}\, nC_{n/2}$.

Also $i^n = (i^2)^{n/2} = (-1)^{n/2}$. & $nC_r = nC_{n-r}$ in (1) we get

①

$$2^n (-1)^{n/2} \sin^n\theta = \left(x^n + \frac{1}{x^n}\right) - nC_1 \left[x^{n-2} + \frac{1}{x^{n-2}}\right] + \dots + (-1)^{n/2} nC_{n/2}$$

$$= (2\cos n\theta) - nC_1 [2\cos(n-1)\theta) + nC_2 (2\cos(n-4)\theta)$$

$$+ \dots + (-1)^{n/2} nC_{n/2}.$$

$$2^n (-1)^n \sin^n\theta = 2\left[\cos n\theta - nC_1 \cos(n-1)\theta + nC_2 \cos(n-4)\theta - \dots + \frac{(-1)^{n/2}}{2} nC_{n/2}\right]$$

$$\therefore \sin^n\theta = \frac{1}{(-1)^{n/2} 2^{n-1}} \left[\cos n\theta - nC_1 \cos(n-2)\theta + nC_2 \cos(n-4)\theta - \dots + \frac{(-1)^{n/2}}{2} nC_{n/2}\right]$$

Step ii) n is odd.

The no. of term in the RHS of ① is even. Therefore the last term is negative $\because (-1)^{odd}$ is $-ve$.
and there are two middle term. they are

$(-1)^{\frac{n-1}{2}} nC_{\frac{n-1}{2}} x$ and $(-1)^{\frac{n+1}{2}} nC_{\frac{n+1}{2}} \left(\frac{1}{x}\right)$. Using $nC_r = nC_{n-r}$ in ① we get

$$2^n i^n \sin^n\theta = \left(x^n - \frac{1}{x^n}\right) - nC_1 \left(x^{n-2} - \frac{1}{x^{n-2}}\right) + \dots + (-1)^{\frac{n-1}{2}} nC_{\frac{n-1}{2}} \left(x - \frac{1}{x}\right).$$

$$= 2i\sin n\theta - nC_1\, 2i\sin(n-2)\theta + nC_2 (2i\sin(n-4)\theta)$$

$$- \dots + (-1)^{\frac{n-1}{2}} nC_{\frac{n-1}{2}} (2i\sin\theta)$$

$$2^n i^n \sin^n\theta = 2i\left[\sin n\theta - nC_1 \sin(n-2)\theta + nC_2 \sin(n-4)\theta - \dots + (-1)^{\frac{n-1}{2}} nC_{\frac{n-1}{2}} \sin\theta\right]$$

$$\sin^n\theta = \frac{1}{2^{n-1} (-1)^{\frac{n-1}{2}}} \left[\sin n\theta - nC_1 \sin(n-2)\theta + nC_2 \sin(n-4)\theta - \dots + (-1)^{\frac{n-1}{2}} nC_{\frac{n-1}{2}} \sin\theta\right]$$

$$\cancel{2}^{n-1}\, \cancel{i}^{n-1}\, 2^1 i^1 = 2i.$$

$$i^{n-1} = \left(i^2\right)^{\frac{n-1}{2}}$$

$$\boxed{i^{n-1} = (-1)^{\frac{n-1}{2}}}$$

Prove that $2^5\cos^6\theta = \cos6\theta + 6\cos4\theta + 15\cos2\theta + 10$. (8)

$$\cos^n\theta = \frac{1}{2^{n-1}}\left[\cos n\theta + nc_1\cos(n-2)\theta + nc_2\cos(n-4)\theta + \cdots\right]$$

$$\cos^6\theta = \frac{1}{2^5}\left[\cos6\theta + 6c_1\cos4\theta + 6c_2\cos2\theta + 6c_3\cos(6-6)\theta\right]$$

$$= \frac{1}{2^5}\left[\cos6\theta + 6\cos4\theta + \frac{6\times5}{1\times2}\cos2\theta + \frac{6\times5\times4}{1\times2\times3}\cos0\right]$$

$$\cos^6\theta = \frac{1}{2^5}\left[\cos6\theta + 6\cos4\theta + 15\cos2\theta + 20\right] \quad (\because \cos0 = 1)$$

$$\therefore 2^5\cos^6\theta = \cos6\theta + 6\cos4\theta + 15\cos2\theta + 20.$$

(or) Another Method

$x = \cos\theta + i\sin\theta$ | $\frac{1}{x} = \cos\theta - i\sin\theta$

$$x + \frac{1}{x} = 2\cos\theta$$

$$(2\cos\theta)^6 = \left(x+\frac{1}{x}\right)^6$$

$$2^6\cos^6\theta = x^6 + 6c_1 x^5\left(\frac{1}{x}\right) + 6c_2 x^4\left(\frac{1}{x}\right)^2 + 6c_3 x^3\left(\frac{1}{x}\right)^3 + 6c_4 x^2\left(\frac{1}{x}\right)^4 + 6c_5 x\left(\frac{1}{x}\right)^5 + 6c_6\left(\frac{1}{x}\right)^6.$$

$$6c_2 = \frac{\overset{3}{\cancel{6}}\times5}{1\times\cancel{2}}$$

$$6c_2 = 15$$

$$= x^6 + 6x^4 + 15x^2 + 20 + 15\left(\frac{1}{x^2}\right) + 6\left(\frac{1}{x^4}\right) + \frac{1}{x^6}$$

$$6c_4 = 6c_2$$

$$6c_3 = \frac{6\times5\times4}{1\times2\times3}$$

$$6c_3 = 20$$

$$6c_5 = 6c_1 = 6$$

$$2^6\cos^6\theta = \left(x^6 + \frac{1}{x^6}\right) + 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) + 20.$$

$$= 2\cos6\theta + 6(2\cos4\theta) + 15(2\cos2\theta) + 20$$

$$2^6\cos^6\theta = 2\cos6\theta + 6\cancel{\times}2\cos4\theta + 15\cos2\theta + 10.$$

$$2^5\cos^6\theta = \cos6\theta + 6\cos4\theta + 15\cos2\theta + 10.$$

Ex 2 Prove that $\sin^5\theta = \frac{1}{2^4}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$ (9)

Solution Let $x = \cos\theta + i\sin\theta$ $\quad \frac{1}{x} = \cos\theta - i\sin\theta$.

$x^n = \cos n\theta + i\sin n\theta$, $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$x - \frac{1}{x} = 2i\sin\theta$.

$x^n - \frac{1}{x^n} = 2i\sin n\theta$.

$(2i\sin\theta)^5 = (x - \frac{1}{x})^5$

$2^5 i^5 \sin^5\theta = x^5 - 5C_1 x^4(\frac{1}{x}) + 5C_2 x^3(\frac{1}{x})^2 - 5C_3 x^2(\frac{1}{x})^3 + 5C_4 x(\frac{1}{x})^4 - 5C_5(\frac{1}{x})^5$

$5C_2 = \frac{5\times 4}{1\times 2} = 10$

$= x^5 - 5x^3 + 10x - 10(\frac{1}{x}) + 5(\frac{1}{x^3}) - \frac{1}{x^5}$

$= (x^5 - \frac{1}{x^5}) - 5(x^3 - \frac{1}{x^3}) + 10(x - \frac{1}{x})$

$5C_3 = 5C_2$

$5C_4 = 5C_1 = 5$

$2^5 i^5 \sin^5\theta = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$

$\sin^5\theta = \frac{2i(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)}{2^5 i^5}$

$\sin^5\theta = \frac{1}{2^4}[\sin 5\theta - 5\sin 3\theta - 10\sin\theta]$

Another Method using formula for $\sin^n\theta$.

$n = 5$ odd.

$\sin^n\theta = \frac{1}{2^{n-1}(-1)^{\frac{n-1}{2}}}[\sin n\theta - nC_1\sin(n-2)\theta + nC_2\sin(n-4)\theta - \dots + (-1)^{\frac{n-1}{2}} nC_{\frac{n-1}{2}}\sin\theta]$

Put $n = 5$

$\sin^5\theta = \frac{1}{2^4}[\sin 5\theta - \overset{5}{5C_1}\sin 3\theta - \overset{10.}{5C_2}\sin\theta]$

Last term

$\frac{5-1}{2} = \frac{4}{2} = 2$

$(-1)^{\frac{n-1}{2}} = +ve$

$5C_{\frac{5-1}{2}} = 5C_2$.

$+5C_2\sin\theta$.

We get the result.

Ex Expand $\cos^5\theta \sin^3\theta$ in a series of sines of multiples of θ. (10)

Soln Let $x = \cos\theta + i\sin\theta$ $\quad \frac{1}{x} = \cos\theta - i\sin\theta$

$x + \frac{1}{x} = 2\cos\theta$, $x - \frac{1}{x} = 2i\sin\theta$.

$$(2\cos\theta)^5 (2i\sin\theta)^3 = (x+\tfrac{1}{x})^5 (x-\tfrac{1}{x})^3$$

$$2^5 \cos^5\theta \, 2^3 i^3 \sin^3\theta = (x+\tfrac{1}{x})^2 \left[(x+\tfrac{1}{x})^3 (x-\tfrac{1}{x})^3\right]$$

$$\Rightarrow i^3 2^8 \cos^5\theta \sin^3\theta = (x+\tfrac{1}{x})^2 \left[(x+\tfrac{1}{x})(x-\tfrac{1}{x})\right]^3$$

$$= (x+\tfrac{1}{x})^2 \left[x^2 - \tfrac{1}{x^2}\right]^3$$

$$= (x^2 + 2 + \tfrac{1}{x^2})\left(x^6 - \tfrac{1}{x^6} - 3x^2 + \tfrac{3}{x^2}\right)$$

$$= x^8 - \frac{1}{x^4} - 3x^4 + 3 + 2x^6 - \frac{2}{x^6} - 6x^2 + \frac{6}{x^2} + x^4 - \frac{1}{x^8} - 3 + \frac{3}{x^4}$$

$$= (x^8 - \tfrac{1}{x^8}) + 2(x^6 - \tfrac{1}{x^6}) - 2(x^4 - \tfrac{1}{x^4}) - 6(x^2 - \tfrac{1}{x^2})$$

$$= 2i\sin 8\theta + 2(2i\sin 6\theta) - 2(2i\sin 4\theta) - 6(2i\sin 2\theta)$$

$$i^3 2^8 \cos^5\theta \sin^3\theta = 2i[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta]$$

$$\cos^5\theta \sin^3\theta = \frac{2i}{2^8 i^3}\left[\quad '' \quad\right]$$

$$\cos^5\theta \sin^3\theta = \frac{1}{-2^7}\left[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right]$$

$\left(\because i^3 = -i, \; i^2 = -1.\right)$

Home Work

Prove the following

(1) $2^3 \cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$

(2) $2^6 \cos^7\phi = \cos 7\phi + 7\cos 5\phi + 21\cos 3\phi + 35\cos\phi$

(3) $2^5 \cos^2\theta \sin^4\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

$f(x) = e^x \quad f'(x) = e^x, \; f''(x) = e^x, \; f'''(x) = e^x, \ldots$

$\therefore f(0) = 1, \quad f'(0) = 1 \quad f''(0) = 1, \ldots$

Using Taylor series expansion of f about origin is

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + \ldots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad \text{—— (1)}$$

Put $x = ix$ in (1) we get

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right).$$

$$\boxed{e^{ix} = \cos x + i\sin x}$$

$\therefore e^{-ix} = \cos x - i\sin x$

$e^{ix} + e^{-ix} = 2\cos x$

$e^{ix} - e^{-ix} = 2i\sin x$

$$\therefore \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

HYPERBOLIC FUNCTIONS

Definitions. The hyperbolic functions are defined by

$$\sinh x = \frac{e^{x}-e^{-x}}{2}, \quad \cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{Cosech} x = \frac{1}{\sinh x}, \quad \operatorname{Sech} x = \frac{1}{\sinh x}.$$

Note:

1. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

2. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \Rightarrow \therefore \cosh x \geq 1$ for all x.

3. $\cosh 0 = 1$ and $\sinh 0 = 0$.

Result 1. $\cosh^2 x - \sinh^2 x = 1$

Proof:

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^{x}+e^{-x}}{2}\right)^2 - \left(\frac{e^{x}-e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x}+e^{-2x}+2e^{x}e^{-x} - e^{+2x} - e^{-2x} + 2e^{x}e^{-x}}{4}$$

$$= \frac{4}{4} = 1 \quad (\because e^{x}e^{-x} = 1).$$

$$\therefore \boxed{\cosh^2 x - \sinh^2 x = 1}$$

2. $\sinh 2x = 2\sinh x \cosh x$.

Proof

$$2\sinh x \cosh x = 2\left(\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$= 2\left(\frac{e^{2x}+e^{x}e^{-x} - e^{-x}e^{x} - e^{-2x}}{4}\right)$$

$$2\sinh x \cosh x = \frac{e^{2x}-e^{-2x}}{2} = \sinh 2x.$$

$$\therefore \boxed{\sinh 2x = 2\sinh x \cosh x}$$

3. $\cosh^2 x + \sinh^2 x = \cosh 2x$.

Proof $\cosh^2 x + \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$

$$= \frac{e^{2x} + e^{-2x} + 2e^x e^{-x}}{4} + \frac{e^{2x} + e^{-2x} - 2e^x e^{-x}}{4}$$

$$= \frac{e^{2x} + e^{-2x} + 2 + e^{2x} + e^{-2x} - 2}{4}$$

$$= \frac{2e^{2x} + 2e^{-2x}}{4} = \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x.$$

$\therefore \cosh 2x = \cosh^2 x + \sinh^2 x$.

(or) $\boxed{\cosh^2 x + \sinh^2 x = \cosh 2x}$

4. From ① & ③ we get the following results.

$\cosh 2x = 2\cosh^2 x - 1$

$\cosh 2x = 1 + 2\sinh^2 x$.

$\left[\because \cosh 2x = \cosh^2 x + \sinh^2 x \right.$
$= \cosh^2 x + \cosh^2 x - 1$
$\left. = 2\cosh^2 x - 1. \right]$

Hence $\cosh^2 x = \frac{\cosh 2x + 1}{2}$

$\sinh^2 x = \frac{\cosh 2x - 1}{2}$

$\cosh 2x = \sinh^2 x + 1 + \sinh^2 x$
$= 2\sinh^2 x + 1$.

Relation between hyperbolic function and circular trigonometric function

Theorem : (i) $\sin(ix) = i\sinh x$
(ii) $\cos(ix) = \cosh x$
(iii) $\tan(ix) = i\tanh x$.

Proof We know that $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$ —①.

Put $\theta = ix$ in ① we get

$$\sin(ix) = (ix) - \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} - \cdots$$

$$= ix - \frac{i^3 x^3}{3!} + \frac{i^5 x^5}{5!} - \cdots$$

$$\sin ix = ix - \frac{(-i)x^3}{3!} + \frac{ix^5}{5!} - \cdots$$

$$= ix + \frac{ix^3}{3!} + \frac{ix^5}{5!} + \cdots$$

$\because i^2 = -1$
$i^3 = i\,i^2 = -i$
$i^4 = 1$
$i^5 = i\,i^4$
$i^5 = i$.

$\sin(ix) = i\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right]$

$\boxed{\sin(ix) = i\sinh x}$

(ii) $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$ ——②

Put $\theta = ix$ in ②, we get

$$\cos(ix) = 1 - \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} - \cdots \qquad \left[\because i^2 = -1,\ i^4 = 1\right]$$

$$= 1 - \frac{i^2x^2}{2!} + \frac{i^4x^4}{4!} - \cdots$$

$$\cos ix = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$\boxed{\cos ix = \cosh x}$

(iii) $\tan(ix) = \frac{\sin(ix)}{\cos(ix)} = \frac{i\sinh x}{\cosh x} = i\tanh x.$

$\therefore \boxed{\tan(ix) = i\tanh x}$

Examples

1. ~~Another~~ Method using $\cos^2\theta + \sin^2\theta = 1$ we get $\cosh^2 x - \sinh^2 x = 1$.

Put $\theta = ix$ in $\cos^2\theta + \sin^2\theta = 1$.

$$\cos^2(ix) + \sin^2(ix) = 1$$

$$(\cos(ix))^2 + (\sin(ix))^2 = 1$$

$$(\cosh x)^2 + (i\sinh x)^2 = 1$$

$$\cosh^2 x + i^2\sinh^2 x = 1 \qquad (\because i^2 = -1)$$

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

2. If $\cos(A+B) = \cos A\cos B - \sin A\sin B$ find $\cosh(x+iy)$.

Soln Put $A = ix$ and $B = iy$ in $\cos(A+B)$

~~we~~ get $\cos(ix+iy) = \cos ix\cos iy - \sin ix\sin iy$

$$\cos(i(x+y)) = \cosh x\cosh y - i\sinh x\, i\sinh y$$

$$\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y \qquad (\because i^2 = -1).$$

14. Write a program to count the occurrence of the character in a string

Another Way

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \Rightarrow \sin(ix) = \frac{e^{i^2 x} - e^{-i^2 x}}{2i}$$

$$= \frac{-i}{2}\left[e^{-x} - e^{x}\right] = i\left[\frac{e^{x} - e^{-x}}{2}\right]$$

$$\sin ix = i\left(\frac{e^{x} - e^{-x}}{2}\right) = i \sinh x.$$